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A Semigroup Perturbation Theorem with Application to a Singular Perturbation Problem in Nonlinear Ordinary Differential Equations

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Let B denote the infinitesimal operator of a strongly continuous semigroup $S(t)$, with resolvent R_λ, on Banach space L. We define related operators P and V so that $\lambda R_\lambda f = Pf + \lambda Vf + o(\lambda)$, as $\lambda \to 0^+$. For $\alpha, \eta > 0$ and possibly unbounded, linear operator A, we let $U_{\alpha,\eta}(t)$ represent a strongly continuous semigroup generated by $\alpha A + \eta B$. We show that under appropriate simultaneous convergence of α and η, $U_{\alpha,\eta}(t)$ converges strongly to a strongly continuous semigroup $U(t)$, having infinitesimal operator characterized through $PA(VA)^r f$ where $r = \min\{j \geqslant 0,\ PA(VA)^j \neq 0\}$. We apply the abstract perturbation theorem to a singular perturbation initial-value problem, of Tihonov-type, for a non-linear system of ordinary differential equations.

INTRODUCTION

In [8] and [9] Thomas G. Kurtz gave abstract limit theorems for semigroups of perturbed infinitesimal generators; he proved these theorems through his extended Trotter-type convergence results given in [8].

We give an extension to these pertubation theorems of the following type. We let B denote the infinitesimal operator of a strongly continuous semigroup $S(t)$, with resolvent R_λ, on Banach space L. We define related operators P and V so that $\lambda R_\lambda f = Pf + \lambda Vf + o(\lambda)$ as $\lambda \to 0^+$. For $\alpha, \eta > 0$ and a possibly unbounded, linear operator A, we let $U_{\alpha,\eta}(t)$ represent a strongly continuous semigroup generated by $\alpha A + \eta B$. We show that under appropriate simultaneous convergence of α and η, $U_{\alpha,\eta}(t)$ converges strongly to a strongly continuous semigroup $U(t)$, having infinitesimal operator characterized through $PA(VA)^r f$, where $r = \min\{j \geqslant 0,\ PA(VA)^j \neq 0\}$. Theorem (1.7) of section one is a perturbation theorem in this framework.

In [10] T. G. Kurtz gives applications of his abstract perturbation theorems to singular perturbation problems for non-linear systems of ordinary differential equations. In section two we apply the abstract theorem of section one to a singular perturbation problem for a non-linear system of ordinary differential

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equations, in the sense of Tihonov [16] and Levin–Levinson [11]. See also O'Malley [12] and [13]. Thus in Theorem (2.15) we show that the solution $(X^\epsilon(t, x, y), Y^\epsilon(t, x, y)) \in R^k \times R^m$, satisfying a non-linear system of o.d.e.'s

$$\begin{aligned}\epsilon^a \dot{X}^\epsilon &= F(X^\epsilon, Y^\epsilon)\\ \epsilon^b \dot{Y}^\epsilon &= G(X^\epsilon, Y^\epsilon) + \epsilon^{b-a}\hat{G}(X^\epsilon, Y^\epsilon)\\ (X^\epsilon(0, x, y), Y^\epsilon(0, x, y)) &= (x, y)\end{aligned}$$

with $a = r(b - a)$ for some non-negative integer r, converges on compact sets as $\epsilon \downarrow 0$ to $(Z(t, x), 0)$ where $Z(t, x)$ satisfies

$$\begin{aligned}\dot{Z}(t, x) &= H(Z(t, x))\\ Z(0, x) &= x,\end{aligned}$$

and H is given explicitly in terms of F, G, and $\hat{G}$ in (2.9). Here, in particular, we assume that the solution $Y_0(t, x, y)$ of the associated boundary-layer equation (2.4) is asymptotically stable. We utilize the semigroup

$$T^\epsilon(t)f = f(X^\epsilon(t, x, y), Y^\epsilon(t, x, y))$$

having infinitesimal generator $\epsilon^{-a}A + \epsilon^{-b}B$ where $Af = F \cdot \mathrm{grad}_x f + \hat{G} \cdot \mathrm{grad}_y f$ and $Bf = G \cdot \mathrm{grad}_y f$, and in the lemmas of section two we verify the conditions of the abstract perturbation theorems of section one. We conclude from Theorem (1.7) of section one that $T_\epsilon(t)$ converges strongly to semigroup $U(t)$, with generator characterized through $J = H \cdot \mathrm{grad}_x f$, and that our singular perturbation result follows.

Theorem (2.15) extends the results in section five of [10].

T. G. Kurtz gives applications of his abstract perturbation theorems to convergence problems for random evolutions. These problems are related to singular perturbation problems for hyperbolic systems of p.d.e.'s and certain abstract Cauchy problems. For surveys of the work in this area see the articles by Hersh [3] and Pinsky [15]. We discuss application to random evolutions of abstract perturbation theorems similar in spirit to those of section one, but with multiply-scaled coefficient functions, in [7].

1

We develop notation and definitions for the abstract perturbation theorem of this section. Let A and B denote linear infinitesimal operators of strongly continuous contraction semigroups $\{T(t): t \geqslant 0\}$ and $\{S(t): t \geqslant 0\}$ respectively, defined on Banach space L. We assume that B is the closure of the operator B restricted to $\mathscr{D}(A) \cap \mathscr{D}(B)$, the intersection of the domains of A and of B, i.e., $\mathscr{D}(A) \cap \mathscr{D}(B)$ is a core of B.

We define operator P by

$$Pf = \lim_{\lambda \to 0^+} \lambda \int_0^\infty e^{-\lambda t} S(t) f \, dt \tag{1.1}$$

with $\mathscr{D}(P)$ given by those f for which this limit exists. We note that if

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T S(t) f \, dt$$

exists, then

$$Pf = \lim_{T \to \infty} \frac{1}{T} \int_0^T S(t) f \, dt; \tag{1.2}$$

and if $\lim_{t \to \infty} S(t) f$ exists, then

$$Pf = \lim_{t \to \infty} S(t) f \tag{1.3}$$

The operator P satisfies

(i) P is a linear, bounded projection;
(ii) $S(t)\, Pf = PS(t)f = Pf$ for all $f \in \mathscr{D}(P)$;
(iii) the range $\mathscr{R}(P)$ of P is the null space $n(B)$ of B; and
(iv) $\mathscr{R}(B)$ is dense in $n(P)$. (1.4)

See Hille–Phillips, ([5], p. 516), for proofs.

We define operator V by

$$Vf = \lim_{\lambda \to 0^+} \int_0^\infty e^{-\lambda t} (S(t) f - Pf) \, dt \tag{1.5}$$

with $\mathscr{D}(V)$ given by those f for which this limit exists. The operator V satisfies

$$\begin{aligned} &\text{(i)} && BVf = Pf - f && \text{for all } f \in \mathscr{D}(V) \\ &\text{(ii)} && VBh = Ph - h && \text{for all } h \in \mathscr{D}(B) \cap \mathscr{D}(P) \\ &\text{(iii)} && BVg = -g && \text{for all } g \in \mathscr{R}(B) \end{aligned} \tag{1.6}$$

THEOREM (1.7). *Given the above notation and definitions. We let* α *and* η *denote positive real parameters satisfying* $\eta \to \infty$ *and* $\alpha^{r+1}/\eta^r \to d$ *for non-negative integer* r *and some* d, $0 < d < \infty$; *this convergence is represented by* $\alpha, \eta \to \Delta$. *We assume for each* α, η *that* $R(\alpha, \eta)$ *is a (possibly unbounded) linear operator such that the closure of* $\alpha A + \eta B + R(\alpha, \eta)$, *with core* $\mathscr{D}(A) \cap \mathscr{D}(B) \cap \mathscr{D}(R(\alpha, \eta))$, *is the infinitesimal operator of a strongly continuous contraction semigroup* $U_{\alpha,\eta}(t)$ *satisfying*

$$\lim_{\alpha,\eta \to \Delta} U_{\alpha,\eta}(t/\eta) f = S(t) f. \tag{1.8}$$

We denote

$$\mathscr{F} = \Big\{ f \in n(B); f \in \mathscr{D}((VA)^j), (VA)^j f \in \mathscr{D}(A) \cap \mathscr{D}(B) \cap \mathscr{D}(R(\alpha, \eta)), \text{ and } A(VA)^j f \in \mathscr{D}(P) \text{ for } j = 0, \ldots, r \text{ with } \lim_{\alpha,\eta \to \Delta} \sum_{j=0}^{r} (\alpha/\eta)^j R(\alpha, \eta)(VA)^j f = 0 \Big\}$$

We assume that $PA(VA)^j f = 0$ *for each* $j = 0, 1, \ldots, r-1$ *and for all* $f \in \mathscr{F}$, *if* $r \geqslant 1$. *We define the operator* C *on* $\mathscr{F}$ *by* $Cf = dPA(VA)^r f$ *and assume that for some* $\lambda > 0$

$$\overline{\mathscr{R}(\lambda I - C)} \supset \mathscr{F}, \tag{1.9}$$

where $\overline{\mathscr{R}(\lambda I - C)}$ *is the closure of set* $\mathscr{R}(\lambda I - C)$.

Then there exists a strongly continuous contraction semigroup $U(t)$ *on* $\overline{\mathscr{F}}$ *with* $\lim_{\alpha,\eta\to\Delta} U_{\alpha,\eta}(t)w = U(t)w$ *for each* $w \in \overline{\mathscr{F}}$. *The infinitesimal operator of* $U(t)$ *is the operator* C *restricted so that* $Cf \in \overline{\mathscr{F}}$.

Proof. From T. G. Kurtz ([9], Theorem 1.10) and ([8], Theorem 2.1) it suffices to exhibit for each $f \in \mathscr{F}$, elements $f_{\alpha,\eta} \in \mathscr{D}(\alpha A + \eta B + R(\alpha, \eta))$ for which $\lim_{\alpha,\eta\to\Delta} f_{\alpha,\eta} = f$ and $\lim_{\alpha,\eta\to\Delta}(\alpha A + \eta B + R(\alpha, \eta)) f_{\alpha,\eta} = Cf$. Given $f \in \mathscr{F}$ we let $h_{\alpha,\eta} = \alpha \sum_{j=1}^{r} (\alpha/\eta)^j (VA)^j f$ and $g_{\alpha,\eta} = \alpha A[f + \alpha^{-1} h_{\alpha,\eta}] + (\eta/\alpha) Bh_{\alpha,\eta}$. Under the given hypotheses there exists for each α and η, $k_{\alpha,\eta} \in \mathscr{D}(A) \cap \mathscr{D}(B) \cap \mathscr{D}(R(\alpha, \eta))$ satisfying $\lim_{\alpha,\eta\to\Delta} \eta^{-1} k_{\alpha,\eta} = 0$, $\lim_{\alpha,\eta\to\Delta} (\alpha/\eta)\, Ak_{\alpha,\eta} = 0$,

$$\lim_{\alpha,\eta\to\Delta} \eta^{-1} R(\alpha, \eta) k_{\alpha,\eta} = 0, \qquad \text{and} \qquad \lim_{\alpha,\eta\to\Delta} Bk_{\alpha,\eta} + g_{\alpha,\eta} = Cf$$

(see [9], Theorem (2.1); [6], p. 31; and the appendix of [7]). We let $f_{\alpha,\eta} = f + \alpha^{-1} h_{\alpha,\eta} + \eta^{-1} k_{\alpha,\eta}$. Then $\lim_{\alpha,\eta\to\Delta} f_{\alpha,\eta} = f$ and since $f \in n(B)$ and from the choice of $\{k_{\alpha,\eta}\}$ we obtain $\lim_{\alpha,\eta\to\Delta}(\alpha A + \eta B + R(\alpha, \eta)) f_{\alpha,\eta} = Cf$. ∎

The special cases of $\alpha = 1$, $\eta = \epsilon^{-1}$ and $\alpha = \epsilon^{-1}$, $\eta = \epsilon^{-2}$ are treated by T. G. Kurtz in ([9], Theorems 2.1 and 2.2). See also the limit theorem of Hersh–Pinsky [4] and the author [6].

For conditions under which the closure of $\alpha A + \eta B + R(\alpha, \eta)$ is the infinitesimal operator of a strongly continuous contraction semigroup, see Pazy [14]. In the case in which $R(\alpha, \eta) = 0$, (1.8) follows from Pazy ([14], Theorem 4.6).

As in T. G. Kurtz ([10], Corollary (2.6)), we can extend the conclusion of Theorem (1.7) to a broader class of functions.

COROLLARY (1.10). *We assume that the definitions and assumptions of Theorem* (1.7) *hold, and that* $f \in \mathscr{D}(P)$ *and* $Pf \in \overline{\mathscr{F}}$.

(i) *For* $\lambda \to +\infty$ *with* $\lambda/\eta \to 0$ *we have for each* $t > 0$

$$\lim_{\alpha,\eta\to\Delta,\lambda\to\infty} \lambda \int_0^\infty e^{-\lambda s} U_{\alpha,\eta}(t+s) f\, ds = U(t)\, Pf. \tag{1.11}$$

(ii) *Assume* (1.2) *holds. Then for* $\delta \to 0$ *with* $(\eta\delta)^{-1} \to 0$ *we have for each* $t > 0$

$$\lim_{\alpha,\eta\to\Delta,\delta\to 0} \delta^{-1} \int_0^{\delta} U_{\alpha,\eta}(t+s) f\, ds = U(t)\, Pf. \tag{1.12}$$

(iii) *Assume* (1.3) *holds. Then for each* $t > 0$ *we have*

$$\lim_{\alpha,\eta\to\Delta} U_{\alpha,\eta}(t) f = U(t)\, Pf. \tag{1.13}$$

We also have for every $t > 0$ *and any* $0 < \delta < 1$

$$\lim_{\alpha,\eta\to\Delta} \sup_{\delta\leqslant t\leqslant\delta^{-1}} \| U_{\alpha,\eta}(t) f - U(t)\, Pf \| = 0 \tag{1.14}$$

We state a lemma which is used in the application of Theorem (1.7) in the following sections. The lemma is used extensively by T. G. Kurtz in ([10], Sections 3, 4, and 5) and is part of "semigroup folklore". We state it here and give a proof for completeness of exposition. For a linear operator C, let $G(C) = \{(f, g); g = Cf\}$ = graph of C and let $C\,|_{\mathscr{D}}$ denote the restriction of C to set $\mathscr{D}$.

LEMMA (1.15). *Suppose* $\mathscr{E}$ *is a Banach space and* $T(t)\colon \mathscr{E} \to \mathscr{E}$ *is a strongly continuous contraction semigroup with infinitesimal operator* $\mathscr{A}$. *Let* $\mathscr{D}$ *be a subset of* $\mathscr{E}$ *with* $\bar{\mathscr{D}} = \mathscr{E}$ *and* $\mathscr{D} \subset \mathscr{D}(\mathscr{A})$. *Suppose* A *is an operator defined on* $\mathscr{D}$ *with* $A = \mathscr{A}\,|_{\mathscr{D}}$ *and* $T(t)\colon \mathscr{D} \to \mathscr{D}(\bar{A})$.

Then $\mathscr{A} = \bar{A}$ *and* $\overline{\mathscr{R}(\lambda I - A)} = \mathscr{E}$.

Proof. We show $\mathscr{R}(\lambda I - \bar{A}) = \mathscr{E}$ and hence $\overline{\mathscr{R}(\lambda I - A)} = \mathscr{E}$. First suppose $g \in \mathscr{D}$ and let $f = \mathscr{R}(\lambda; \mathscr{A})g = \int_0^\infty e^{-\lambda t} T(t) g\, dt$ and $f_n = n^{-1} \sum_{i=0}^{n^2} e^{-\lambda i/n} T(i/n) g$. We have $\lim_n f_n = f$ and $\lim_n \bar{A} f_n = \int_0^\infty e^{-\lambda t} T(t) A g\, dt$; thus $f \in \mathscr{D}(\bar{A})$ and $\lambda f - \bar{A} f = \lambda f - \mathscr{A} f = g$. Now suppose $g \in \mathscr{E}$; there exist $\{g_n\} \subset \mathscr{D}$ with $\lim_n g_n = g$. We have $f_n \equiv \mathscr{R}(\lambda; \mathscr{A}) g_n \in \mathscr{D}(\bar{A})$, $\lambda f_n - \bar{A} f_n = g_n$, and $\lim_n f_n = \mathscr{R}(\lambda; \mathscr{A}) g \equiv f$. Thus $\lim_n \bar{A} f_n = \lim_n \lambda f_n - g_n = \lambda f - g$, $f \in \mathscr{D}(\bar{A})$, and $\lambda f - \bar{A} f = g$.

Because we also have that $\mathscr{D}(\bar{A})$ is dense in $\mathscr{E}$ and $\| \lambda f - \bar{A} f \| \geqslant \lambda \| f \|$ for each $\lambda > 0$ and all $f \in \mathscr{D}(\bar{A})$, we can apply the Hille–Yosida Theorem ([2], p. 30) to obtain that $\bar{A}$ is the infinitesimal operator of a strongly continuous contraction semigroup $U(t)\colon \mathscr{E} \to \mathscr{E}$. If we denote $\mathscr{R}(\lambda; \bar{A}) g = \int_0^\infty e^{-\lambda t} U(t) g\, dt$, then $(\lambda - \bar{A})\, \mathscr{R}(\lambda; \bar{A}) g = g$ for each $g \in \mathscr{E}$. Because $\mathscr{R}(\lambda; \bar{A})\, g \in \mathscr{D}(\bar{A}) \subset \mathscr{D}(\mathscr{A})$, we have $(\lambda - \mathscr{A})\, \mathscr{R}(\lambda; \bar{A}) g = g$ and it follows that $\mathscr{R}(\lambda; \bar{A}) g = \mathscr{R}(\lambda; \mathscr{A}) g$. From ([2], Lemma 1.1, p. 26), we obtain $T(t) = U(t)$ and $\mathscr{A} = \bar{A}$. ∎

2

In this section we apply Theorem (1.7) of Section 1 to a singular perturbation initial value problem for a nonlinear system of ordinary differential equations.

We let $F: R^{k+m} \times (0, \infty) \to R^k$ and $G: R^{k+m} \times (0, \infty) \to R^m$ denote mappings with growth and regularity conditions given in the hypotheses (H_1)–(H_7) below. Let $\epsilon > 0$ and $b > a > 0$ be real parameters and r a positive integer with $a = r(b - a)$. We consider a singular perturbation initial value problem for $(X^\epsilon(t, x, y), Y^\epsilon(t, x, y)) \in R^k \times R^m$ satisfying the non-linear system of ordinary differential equations

$$\begin{aligned} \epsilon^a \dot{X}^\epsilon &= F(X^\epsilon, Y^\epsilon, \epsilon) \\ \epsilon^b \dot{Y}^\epsilon &= G(X^\epsilon, Y^\epsilon, \epsilon) \\ (X^\epsilon(0, x, y), Y^\epsilon(0, x, y)) &= (x, y) \end{aligned} \tag{2.1}$$

Equivalently, we have, with $(X_\epsilon(t, x, y), Y_\epsilon(t, x, y))$ satisfying

$$\begin{aligned} \dot{X}_\epsilon &= \epsilon^{b-a} F(X_\epsilon, Y_\epsilon, \epsilon) \\ \dot{Y}_\epsilon &= G(X_\epsilon, Y_\epsilon, \epsilon) \\ (X_\epsilon(0, x, y), Y_\epsilon(0, x, y)) &= (x, y), \end{aligned} \tag{2.2}$$

that $(X_\epsilon(t/\epsilon^b), Y_\epsilon(t/\epsilon^b))$ satisfies (2.1). Here we consider the special case

$$\begin{aligned} F(x, y, \epsilon) &= F(x, y) + \epsilon^a \hat{F}(x, y, \epsilon) \\ G(x, y, \epsilon) &= G(x, y) + \epsilon^{b-a} \hat{G}(x, y) + \epsilon^b \hat{\hat{G}}(x, y, \epsilon). \end{aligned} \tag{2.3}$$

Associated with system (2.1) are two auxiliary systems. The "boundary-layer" equation is given by

$$\begin{aligned} \dot{X}_0 &= 0 \\ \dot{Y}_0 &= G(X_0, Y_0) \\ (X_0(0, x, y), Y_0(0, x, y)) &= (x, y); \end{aligned} \tag{2.4}$$

the reduced equation is given by

$$\begin{aligned} 0 &= F(X, Y) \\ 0 &= G(X, Y) \\ (X(0, x, y), Y(0, x, y)) &= (x, y). \end{aligned} \tag{2.5}$$

We list the hypotheses utilized in the proof of our perturbation theorem. For notational convenience we let $\partial_y G(x, y)$ denote the matrix $((\partial/\partial y_i)\, G_j(x, y))$, and $\partial^r_{y_{i_1} \cdots y_{i_r}} F(x, y)$ denote the function $(\partial^r F(x, y)/\partial y_{i_1} \cdots \partial y_{i_r})$.

(H_1) Functions F, $\hat{F}$, G, $\hat{G}$, $\hat{\hat{G}}$ are Lipschitz continuous.

(H_2) First through rth partials of F, G, and $\hat{G}$ exist and are Lipschitz continuous (see (H_7)).

(H_3) Y_0 is asymptotically stable at zero, i.e.,

$$\lim_{t\to\infty} \sup_x \sup_{y \in C} | Y_0(t, x, y)| = 0 \tag{2.6}$$

holds for each compact set C; and there is some $\mu > 0$ such that the eigenvalues of $\partial_y G(x, 0)$ have real parts less than $-\mu$.

(H_4) Functions $\hat{G}(x, 0)$, $(\partial_y G(x, 0))^{-1}$, and $\partial^r_{y_{i_1} \cdots y_{i_r}} F(x, 0)$, together with their first through $(r + 1)$st partials, exist and are Lipschitz continuous and bounded. One of the functions $(\partial_{y_{i_1} \cdots y_{i_r}} F(x, 0))$, $\hat{G}(x, 0)$ or $(\partial_y G(x, 0))^{-1}$ goes to zero as x goes to infinity.

(H_5) Given $\eta > 0$, there exists ϵ_0 sufficiently small so that $\epsilon < \epsilon_0$ implies

$$\sup_x \sup_y |\hat{F}(x, y, \epsilon)| \leqslant \eta$$
$$\sup_x \sup_y |\hat{\hat{G}}(x, y, \epsilon)| \leqslant \eta.$$

(H_6) We have $F(x, 0) = 0 = \partial^n_{y_{i_1} \cdots y_{i_n}} F(x, 0)$, $n = 1, \ldots, r - 1$.

(H_7) There exists an $M > 0$ such that for $|y| > M$, we have $F(x, y, \epsilon) = 0$ and $\hat{G}(x, y) = 0 = \hat{\hat{G}}(x, y, \epsilon)$ for each $\epsilon > 0$.

We comment on these hypotheses. From (H_3) we have that $G(x, 0) = 0$. Since we are interested in a "local" result, (H_7) can be imposed on F, $\hat{G}$, and $\hat{\hat{G}}$ satisfying (H_1)–(H_6) through a modification which does not add further restrictions to our theorem; similarly other of the hypotheses can be added through modifications. Through (H_7) we have that there exists $M > 0$ for which

$$\sup_{|y| \leqslant M} \sup_x \sup_{t > 0} |Y_0(t, x, y)| = M_1 < \infty \tag{2.7}$$

and if $|y| > M_1$, then $|Y_0(-t, x, y)| > M$ for all $t > 0$ and hence

$$Y_0(-t, x, y) = Y_\epsilon(-t, x, y) \tag{2.8}$$

for all $t > 0$.

There is some overlap here: e.g., given that $(\partial_y G(x, 0))^{-1}$ is bounded, (H_2) gives that $(\partial_y G(x, 0))^{-1}$ is Lipschitz continuous; and (H_7) allows less restrictive versions of (H_2) and (H_5).

We now set up the system satisfied by the limiting solution of our perturbation problem. We define $H: R^k \to R^k$ by

$$\begin{aligned} H(x) &= \sum_{1 \leqslant j_1, \ldots, j_r \leqslant m} c_{j_1, \ldots, j_r}(x)\, D_{2,j_1} \cdots D_{2,j_r} F(x, 0) \\ &= \frac{1}{r!} \sum_{1 \leqslant j_1, \ldots, j_r \leqslant m} a_{j_1}(x) \cdots a_{j_r}(x)\, D_{2,j_1} \cdots D_{2,j_r} F(x, 0) \\ &= \frac{1}{r!} \left(\sum_{j=1}^m a_j(x)\, D_{2,j} \right)^r F(x, 0) \\ &= \frac{1}{r!} (\hat{G}(x, 0)\, R(x)\, D_2)^r F(x, 0) \\ &= \frac{(-1)^r}{r!} (\hat{G}(x, 0)(\partial_y G(x, 0))^{-1} D_2)^r F(x, 0) \end{aligned} \tag{2.9}$$

where $D_{2,j}f(x, y) = \partial_{y_j}f(x, y)$ and $D_2 = (D_{2,1}, \dots, D_{2,m})$;

$$c_{j_1,\dots,j_r}(x) = \int_0^\infty \cdots \int_0^\infty J_{j_1}\left(\sum_{i=1}^r t_i, x\right) \cdots J_{j_r}(t_r, x)\, dt_1 \cdots dt_r \tag{2.10}$$

with

$$J_\alpha(t, x) = \sum_{i=1}^m \hat{G}_i(x, 0)(\exp\{t\partial_y G(x, 0)\})_{i,\alpha};$$

$$a_j(x) = \sum_{i=1}^m \hat{G}_i(x, 0) \int_0^\infty \{\exp t\partial_y G(x, 0)\}_{i,j}\, dt$$

$$= (-1) \sum_{i=1}^m \hat{G}_i(x, 0)(\partial_y G(x, 0))_{i,j}^{-1}; \quad \text{and} \tag{2.11}$$

$$R(x) = \int_0^\infty \exp\{t\partial_y G(x, 0)\}\, dt. \tag{2.12}$$

From hypotheses (H_1), (H_2), and (H_4) we obtain that each $a_j(x)$ and $H(x)$ is Lipschitz continuous and bounded. From boundedness of each $a_j(x)$ and $\partial^r_{y_{i_1}\cdots y_{i_r}}F(x, 0)$, together with ($H_6$), we obtain $\lim_{|x|\to\infty} | H(x)| = 0$. We denote by $Z(t, x)$ the solution of the nonlinear system

$$\begin{aligned} \dot{Z}(t, x) &= H(Z(t, x)) \\ Z(0, x) &= x \end{aligned} \tag{2.13}$$

From Lipschitz continuity of H we have that (2.13) has a unique solution satisfying

$$\lim_{|x_0|+\tau\to 0} \sup_{0\leqslant t\leqslant\tau} \sup_{|x|\leqslant|x_0|} | Z(t, x) - x | = 0. \tag{2.14}$$

The following theorem extends Theorem 5.11, T. G. Kurtz, [10].

THEOREM (2.15). *Given the above definitions and under hypotheses* (H_1)–(H_7). *If* $(X_\epsilon(t, x, y), Y_\epsilon(t, x, y))$ *denotes the solution to* (2.2) *and* $Z(t, x)$ *denotes the solution of* (2.13), *we have for each* $\delta > 0$ *and each compact set* C *in* R^{k+m}

$$\lim_{\epsilon\to 0} \sup_{0\leqslant t\leqslant\delta} \sup_{(x,y)\in C} | X_\epsilon(t/\epsilon^b, x, y) - Z(t, x)| = 0 \tag{2.16}$$

$$\lim_{\epsilon\to 0} \sup_{1/\delta\leqslant t\leqslant\delta} \sup_{(x,y)\in C} | Y_\epsilon(t/\epsilon^b, x, y)| = 0 \tag{2.17}$$

Proof. We prove that this theorem follows from Theorem (1.7). We first give the functional-analysis context for this application. We then verify that conditions (1.3) and (1.8) hold in this context by proving Lemma (2.26). Next we introduce sets and operators which allow us to work with the desired limiting infinitesimal operator. We then prove that Condition (1.9) holds here through Lemmas (2.29) and (2.32). Finally, we finish the proof by applying Theorem (1.7) and by showing that its conclusion implies (2.16) and (2.17).

Since in (2.6) the asymptotic behavior of $Y_0(t, x, y)$ as $t \to \infty$ is not uniform in all of the y space, we introduce the norming function γ: $R^{k+m} \to R$ by

$$\gamma(x, y) = \sup_{t>0} \Gamma(|\, Y_0(-t, x, y)|) = \sup_{t>0} \Gamma(|\, Y_\epsilon(-t, x, y)|)$$

where $\Gamma: R \to (0, 1]$ is any continuously differentiable function satisfying $\Gamma(u) > 0$, $\Gamma'(u) \leqslant 0$, $\Gamma(u) = 1$ for $u < M_1$, and $\lim_{u\to\infty} \Gamma(u) = 0$. The function γ satisfies the following properties:

$$\begin{aligned}
&\text{(i)} \quad 0 < \gamma(x, y) \leqslant 1 \\
&\text{(ii)} \quad \lim_{|y|\to\infty} \sup_x \gamma(x, y) = 0 \\
&\text{(iii)} \quad \lim_{t\to 0} \frac{1}{t} (\gamma(x, Y_0(t, x, y)) - \gamma(x, y)) \geqslant 0 \\
&\text{(iv)} \quad \lim_{t\to 0} \frac{1}{t} (\gamma(X_\epsilon(t, x, y), Y_\epsilon(t, x, y)) - \gamma(x, y)) \geqslant 0
\end{aligned} \tag{2.18}$$

With notation $E = R^{k+m}$, we introduce the function spaces $\hat{C}(E)$ = continuous functions on E vanishing at infinity;

$C_\gamma(E)$ = closure of $\hat{C}(E)$ with respect to $\|\cdot\|_\gamma$, where $\|f\|_\gamma = \sup_{x,y} |\, \gamma(x, y) f(x, y)|$; and

$C_0^n(E)$ = continuous functions on E with compact support, having jth derivative continuous, $j = 1,\ldots, n$

We define operators B_0, A_0, $R_0(\epsilon)$, and $C_0(\epsilon)$ for $\epsilon > 0$ and $f \in C_0^1(E)$ by

$$B_0 f(x, y) = G(x, y) \cdot \operatorname{grad}_y f(x, y) \tag{2.19}$$

$$A_0 f(x, y) = F(x, y) \cdot \operatorname{grad}_x f(x, y) + \hat{G}(x, y) \cdot \operatorname{grad}_y f(x, y) \tag{2.20}$$

$$\begin{aligned} R_0(\epsilon) f(x, y) = \epsilon^b\{&\hat{F}(x, y, \epsilon) \cdot \operatorname{grad}_x f(x, y) \\ &+ \hat{G}(x, y, \epsilon) \cdot \operatorname{grad}_y f(x, y)\} \end{aligned} \tag{2.21}$$

$$C_0(\epsilon) f(x, y) = B_0 f(x, y) + \epsilon^{b-a} A_0 f(x, y) + R_0(\epsilon) f(x, y) \tag{2.22}$$

We denote by B, A, $R(\epsilon)$, and $C(\epsilon)$ the closure in $C_\gamma(E)$ respectively of B_0, A_0, $R_0(\epsilon)$, and $C_0(\epsilon)$, each $\epsilon > 0$.

Since γ satisfies (2.18)(iii), (iv) we have by Corollary 1.7 of [10] that B and $C(\epsilon)$ generate strongly continuous contraction semigroups on $C_\gamma(E)$, respectively given by $\{S(t), t \geqslant 0\}$ and $\{T_\epsilon(t), t \geqslant 0\}$, for each $\epsilon > 0$. For each $f \in \hat{C}(E)$ we have

$$S(t) f(x, y) = f(x, Y_0(t, x, y)) \tag{2.23}$$

where Y_0 is the solution of (2.4), and

$$T_\epsilon(t) f(x, y) = f(X_\epsilon(t, x, y), Y_\epsilon(t, x, y)) \tag{2.24}$$

where (X_ϵ, Y_ϵ) is the solution of (2.2); also $S(t) |_{\hat{C}(E)}$ is a group generated by the closure of B_0 in $\hat{C}(E)$ with respect to the supremum norm $\| \cdot \|$, and $T_\epsilon(t) |_{\hat{C}(E)}$ is a group generated by the closure of $C_0(\epsilon)$ in $\hat{C}(E)$ with respect to $\| \cdot \|$ (See Theorem 1.1 of [10]).

Our interest is in the behavior of $T_\epsilon(t/\epsilon^b)$ as $\epsilon \to 0$; in particular, for $f \in \hat{C}(E)$ we study the behavior of $f(X_\epsilon(t/\epsilon^b, x, y), Y_\epsilon(t/\epsilon^b, x, y))$ as $\epsilon \to 0$. We observe that $\{T_\epsilon(t/\epsilon^b), t \geqslant 0\}$ has infinitesimal generator $\epsilon^{-b}C(\epsilon) = \epsilon^{-a}A + \epsilon^{-b}B + \epsilon^{-b}R(\epsilon)$.

We define the projection operator P on $C_\gamma(E)$ by

$$Pf(x, y) = f(x, 0). \tag{2.25}$$

The following lemma shows that this definition of P is consistent with defining P through (1.3).

Lemma (2.26). *For each $f \in C_\gamma(E)$, we have that $\lim_{t\to\infty} S(t)f = Pf$ and $\lim_{\epsilon\to 0} T_\epsilon(t)f = S(t)f$.*

Proof. It suffices to prove these results for $f \in \hat{C}(E)$ since $\hat{C}(E)$ is dense in $C_\gamma(E)$. To prove that $\lim_{t\to\infty} S(t)f = Pf$ for $f \in \hat{C}(E)$, from (2.23) and (2.25) we must show that given $\epsilon > 0$, there exists $T > 0$ so that for all $t > T$

$$\sup_{x,y} | \gamma(x, y)[f(x, Y_0(t, x, y)) - f(x, 0)]| < \epsilon.$$

From (2.18ii), and since f vanishes at infinity, it suffices to show for N_0, M, and δ given, that there exists $T > 0$ so that for all $t > T$,

$$\sup_{|x|\leqslant N_0} \sup_{|y|\leqslant M} | Y_0(t, x, y)| < \delta.$$

This follows from (2.6).

We now prove that $\lim_{\epsilon\to 0} T_\epsilon(t)f = S(t)f$ for $f \in \hat{C}(E)$. From the usual continuous dependence argument for systems of o.d.e.'s depending upon a parameter, we have for given parameters $\eta > 0$ and $\delta > 0$ and compact set K in E that there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$,

$$\sup_{0\leqslant t\leqslant \delta} \sup_{(x,y)\in K} | W_\epsilon(t, x, y) - W_0(t, x, y)| < \eta \tag{2.28}$$

where $W_\epsilon(t, x, y) = (X_\epsilon(t, x, y), Y_\epsilon(t, x, y))$ is the solution of (2.2) and $\dot{W}_0(t, x, y) = (x, Y_0(t, x, y))$ is the solution of (2.4). Given $\eta > 0$, there thus exists M, N, and ϵ_0 such that for all $0 < \epsilon < \epsilon_0$

$$\begin{aligned} \| T_\epsilon(t) f - S(t) f \|_\gamma &= \sup_{x,y} | \gamma(x, y)[f(W_\epsilon(t, x, y)) - f(W_0(t, x, y))]| \\ &\leqslant 2\eta + \sup_{|x| \leqslant N} \sup_{|y| \leqslant M} | \gamma(x, y)[f(W_\epsilon(t, x, y)) - f(W_0(t, x, y))]| \\ &\leqslant 3\eta. \end{aligned}$$

Here we have used (2.23), (2.24), (2.18ii), (2.28), and properties of $f \in \hat{C}(E)$. ▌

Using $S(t)$ and P given above, we define operator V through (1.5). It follows from (H_3) that for $f \in C_0^1(E)$ we have

$$Vf(x, y) = \int_0^\infty [f(x, Y_0(t, x, y)) - f(x, 0)]\, dt.$$

We denote $\mathscr{D} = \{f \in C_\gamma(E);\ f(x, y) = f(x, 0) = f(x), f \in C^{r+1}(E),\ x \to f(x, 0)$ has compact support in $R^k\}$.

By the natural identification, we think of $\mathscr{D}$ as $C_0^{r+1}(R^k)$.

Lemma (2.29). *For each* $f \in \mathscr{D}$, $PA(VA)^j f = 0$ *for* $j = 0, 1,\dots, r - 1$, *and* $PA(VA)^r f(x) = H(x) \cdot \operatorname{grad}_x f(x)$, *where* H *is given in* (2.9).

Proof. For $f \in \mathscr{D}$ we have from (H_7) that $f \in \mathscr{D}(A)$ and $Af(x, y) = 0$ for $| y | > M$. Now, for any function g in $C_\gamma(E)$ for which there exists K so that $g(x, y) = 0$ for all $| y | > K$, we have from (H_1), (H_2), (H_3), (H_4), (H_6), and (H_7) that $g \in \mathscr{D}(V)$. Hence $Af \in \mathscr{D}(V)$. Again, using (H_1), (H_2), (H_3), (H_6), and (H_7), we obtain that $f \in \mathscr{D}((VA)^j)$ and $(VA)^j f \in \mathscr{D}(A)$ for $j = 2,\dots, r$. It is clear from (H_6) that $PA(VA)^j f = 0$ for $j = 0, 1,\dots, r - 1$; finally, we have

$$\begin{aligned} PA(VA)^r f(x, y) &= A(VA)^r f(x, 0) \\ &= \sum_{i=1}^{m} (\hat{G}(x, 0))_i\, D_{2,i}(VA)^r f(x, 0) \\ &= \sum_{1 \leqslant v,u,s \leqslant m} (\hat{G}(x, 0)_v (K_1(x))_{v,u} D_{2,u} D_{2,s}(VA)^{r-1} f(x, 0)(\hat{G}(x, 0))_s \\ &= \sum_{\substack{1 \leqslant i_n, \alpha_n, j_n \leqslant m \\ 1 \leqslant p \leqslant k}} (K_1(x))_{i_1, j_1} (K_2(x))_{j_1, j_2 / i_2, j_3} \cdots \\ &\qquad (K_r(x))_{j_{m+1}, \alpha_1 / j_{m+2}, \alpha_2 / \cdots / j_{m+(r+1)}, \alpha_{r-1} / i_r, \alpha_r} \\ &\qquad D_{2,\alpha_1} \cdots D_{2,\alpha_r} F_p(x, 0)\, D_{1,p} f(x, 0)(\hat{G}(x, 0))_{i_1} \cdots (\hat{G}(x, 0))_{i_r} \end{aligned}$$

$$= \sum_{\substack{1 \leqslant i_n, \alpha_n \leqslant m \\ 1 \leqslant p \leqslant k}} \left\{ \int_0^\infty \cdots \int_0^\infty \left(\exp\left(\sum_{n=1}^r t_n \right) \partial_y G(x, 0) \right)_{i_1, \alpha_1} \cdots \right.$$

$$(\exp t_r\, \partial_y G(x, 0))_{i_r, \alpha_r}\, dt_1 \cdots dt_r$$

$$\left. D_{2,\alpha_1} \cdots D_{2,\alpha_r} F_p(x, 0)\, D_{1,p} f(x, 0)(\hat{G}(x, 0))_{i_1} \cdots (\hat{G}(x, 0))_{i_r} \right\}$$

$$= H(x) \cdot \operatorname{grad}_x f(x, 0)$$

$$= Jf(x, y)$$

where

$$(K_1(x))_{i_1, j_1} = \int_0^\infty (\exp t\, \partial_y G(x, 0))_{i_1, j_1}\, dt$$

$$(K_2(x))_{j_1, j_2 / i_2, j_3} = \int_0^\infty (\exp t\, \partial_y G(x, 0))_{j_1, j_2} (\exp t\, \partial_y G(x, 0))_{i_2, j_3}\, dt$$

$$(K_r(x))_{j_{m+1}, \alpha_1 / j_{m+2}, \alpha_2 / \cdots / j_{m+(r-1)}, \alpha_{r-1} / i_r, \alpha_r} = \int_0^\infty (\exp t\, \partial_y G(x, 0))_{j_{m+1}, \alpha_1} \cdots (\exp t\, \partial_y G(x, 0))_{i_r, \alpha_r}\, dt$$

and $H(x)$ is given in (2.9). ∎

We define the operator J_0 for $g \in C_0{}^1(E)$ by

$$J_0 g(x, y) = H(x) \cdot \operatorname{grad}_x g(x, 0) \tag{2.30}$$

where H is given in (2.9). Let J denote the closure of J_0 in the C_γ-norm. From From Corollary 1.7 of [10] we have that J is the infinitesimal operator of a strongly continuous contraction semigroup $T(t)$ on $C_\gamma(E)$. Since $J|_{\hat{C}(E)}$ is the closure of J_0 in the sup norm on $\hat{C}(E)$, it follows from Theorem 1 of [10] that

$$T(t) f(x, y) = f(Z(t, x), 0) \tag{2.31}$$

for all $f \in \hat{C}(E)$.

LEMMA (2.32). *For $f \in \mathscr{D}$, we have $T(t) f \in \mathscr{D}$ and* (2.31) *holds.*

Proof. From $\|\cdot\|_\gamma$-approximation of $f \in \mathscr{D}$ by $\{g_n\} \subset \hat{C}(E)$, with $T(t) g_n(x, y) = g_n(Z(t, x), 0)$, together with limited growth of $Z(t, x)$ due to H being bounded and continuous, we have (2.31) holds for $f \in \mathscr{D}$. We show $T(t)\colon \mathscr{D} \to \mathscr{D}$. Assumption ($H_3$) together with Theorem 7 of Coppel ([1], p. 24) give that $\partial^n_{x_{j_1} \cdots x_{j_n}} Z(t, x)$, $1 \leqslant n \leqslant r + 1$, are continuous and hence that $T(t) f \in C^{r+1}(E)$. That the mapping $x \to T(t)f(x, y) = f(Z(t, x), 0)$ has compact support follows from the continuous dependence on initial conditions for the solution $Z(t, x)$ of (2.13). ∎

Conclusion of the Proof of Theorem (2.15): As in Section one we denote

$$\begin{aligned}\mathscr{F} = \{&f \in n(B); f \in \mathscr{D}((VA)^j) \text{ and } (VA)^j f \in \mathscr{D}(A) \\ &\text{for } j = 0, 1,..., r, \text{ and } (VA)^j f \in \mathscr{D}(R(\epsilon)) \cap \mathscr{D}(B) \text{ for} \\ &\text{each } \epsilon > 0, \text{ with } \lim_{\epsilon \to 0} \epsilon^{-(b-j(b-a))} R(\epsilon)(VA)^j f = 0 \\ &\text{for } j = 1,..., r\}.\end{aligned}$$

From Lemma (2.29) and Assumptions (H_5) and (H_6), we obtain $\mathscr{D} \subset \mathscr{F}$; it is also clear that the closure of $\mathscr{D}$ within $\bar{\mathscr{F}}$ is $\bar{\mathscr{F}}$. From Lemmas (2.29) and (2.32), we can apply Lemma (1.15) to obtain $\overline{\mathscr{R}(\lambda I - PA(VA)^r)} \supset \mathscr{F}$. From Theorem (1.7) we conclude that $\overline{PA(VA)^r}$ generates a strongly continuous contraction semigroup $\{U(t), t \geqslant 0\}$, defined on $\bar{\mathscr{F}}$, satisfying

$$\lim_{\epsilon \to 0} \sup_{0 \leqslant t \leqslant 1/\delta} \| T_\epsilon(t/\epsilon^b) f - U(t) f \|_\gamma = 0 \tag{2.33}$$

for each $f \in \bar{\mathscr{F}}$ and $\delta > 0$. From Corollary (1.10) we conclude that

$$\lim_{\epsilon \to 0} \sup_{\delta < t < 1/\delta} \| T_\epsilon(t/\epsilon^b) f - U(t)\, Pf \|_\gamma = 0 \tag{2.34}$$

for each $f \in C_\gamma(E)$ for which $Pf \in \bar{\mathscr{F}}$, for each $0 < \delta < 1$.

We finish the proof of the theorem by showing that (2.33) implies (2.16), and (2.34) implies (2.17). We use the following special forms of (2.33) and (2.34). For each compact set C in E, $\delta > 0$, and $f \in \mathscr{D}$ we have

$$\lim_{\epsilon \to 0} \sup_{0 \leqslant t \leqslant 1/\delta} \sup_{(x,y) \in C} | f(X_\epsilon(t/\epsilon^b, x, y), Y_\epsilon(t/\epsilon^b, x, y)) - f(Z(t, x), 0)| = 0; \tag{2.35}$$

and for each $f \in \mathscr{D}$, each C compact in E, and $0 < \delta < 1$, we have from (2.34)

$$\lim_{\epsilon \to 0} \sup_{\delta \leqslant t \leqslant 1/\delta} \sup_{(x,y) \in C} | f(X_\epsilon(t/\epsilon^b, x, y), Y_\epsilon(t/\epsilon^b, x, y)) - f(Z(t, x), 0))| = 0. \tag{2.36}$$

Suppose (2.16) does not hold. Then for some $\eta > 0$, there exists $\epsilon_n \downarrow 0$, $\delta > 0$, and C compact for which

$$\sup_{0 \leqslant t \leqslant 1/\delta} \sup_{(x,y) \in C} | X_{\epsilon_n}(t/\epsilon_n{}^b, x, y) - Z(t, x)| > \eta.$$

For this $\eta > 0$ we have for $n \geqslant 1$, $\epsilon_n \downarrow 0$, $0 \leqslant t_n \leqslant 1/\delta$, and $(x_n, y_n) \in C$ satisfying

$$| X_{\epsilon_n}(t/\epsilon_n{}^b, x_n, y_n) - Z(t_n, x_n)| > \eta. \tag{2.37}$$

Let $\tilde{C} = \bigcup_{n=1}^{\infty}\{Z(t_n, x_n)\}$ and $\{E_j\}_{j=1,\ldots,s}$ is a finite $\eta/2$-partition of $\tilde{C}$. Let f_j be any function in $\mathscr{D}$ for which

$$
\begin{aligned}
f_j(x, y) &= a_j \qquad \text{if} \quad x \in E_j \\
&= 0 \qquad \text{if} \quad x \notin \eta/4 - \text{neighborhood of } E_j
\end{aligned}
$$

where $| a_j - a_k | \geqslant 1$, $j \neq k$. Then for $f = \sum_{j=1}^{s} f_j \in \mathscr{D}$ and these $\{\epsilon_n\}$, $\{t_n\}$, $\{(x_n, y_n)\}$ we have from (2.37) that

$$
| f(X_{\epsilon_n}(t_n/\epsilon_n{}^b, x_n, y_n), Y_{\epsilon_n}(t_n/\epsilon_n{}^b, x_n, y_n)) - f(Z(t_n, x_n), 0)| \geqslant 1
$$

for all $n \geqslant 1$. This contradicts (2.35). Thus (2.16) must hold. Similarly, if (2.17) does not hold, then (2.36) fails to hold; thus (2.17) must hold. The proof of Theorem (2.15) is complete. ∎

Remarks. (i) Of course, the limiting solution $(Z(t, x), 0)$ satisfies the reduced system (2.5) since $F(x, 0) = 0 = G(x, 0)$ from (H_3) and (H_6).

(ii) As a simple example consider functions F, $\hat{G}: R^2 \to R$ with sufficient regularity and growth properties so that (H_1)–(H_4) are satisfied and with $0 = F(x, 0) = (\partial/\partial y) F(x, 0) = \cdots = (\partial^{r-1}/\partial y^{r-1}) F(x, 0)$. Given b, a with $b > a > 0$ and $a = r(b - a)$. Let $(X^\epsilon(t, x, y), Y^\epsilon(t, x, y))$ satisfy

$$
\begin{aligned}
\epsilon^a \dot{X}^\epsilon &= F(X^\epsilon, Y^\epsilon) \\
\epsilon^b \dot{Y}^\epsilon &= -Y^\epsilon + \epsilon^{b-a}\hat{G}(X^\epsilon, Y^\epsilon) \\
(X^\epsilon(0, x, y), Y^\epsilon(0, x, y)) &= (x, y)
\end{aligned}
\tag{2.43}
$$

and let $(Z(t, x))$ satisfy

$$
\begin{aligned}
\dot{Z} &= \frac{1}{r!} [\hat{G}(Z, 0)]^r \frac{\partial^r F}{\partial y^r} (Z, 0) \\
Z(0, x) &= x
\end{aligned}
\tag{2.44}
$$

Then from Theorem (2.15) we conclude

$$
\lim_{\epsilon \to 0} \sup_{0 \leqslant t \leqslant 1/\delta} \sup_{(x,y) \in C} | X^\epsilon(t, x, y) - Z(t, x)| = 0
$$

for each $\delta > 0$, C compact in R^2, and

$$
\lim_{\epsilon \to 0} \sup_{\delta \leqslant t \leqslant 1/\delta} \sup_{(x,y) \in C} | Y^\epsilon(t, x, y)| = 0
$$

for each $0 < \delta < 1$, C compact in R^2.

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